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The symmetry lessons from Froebel building gifts

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Abstract. The symmetry properties of the Froebel building gifts are discussed in detail. A special emphasis is given to the cycle indices of their permutation groups. They are used in the enumeration of $n \leqslant 3$ colour assignments on the faces of the building blocks.

1 Introduction

The *kindergarten grammars* (Stiny, 1980) have nicely captured the pedagogical character of the *kindergarten method* of Frederick Froebel (Brosterman, 1997). The division of Froebel's kindergarten method into a series of geometrical *gifts* and a system of *categories* of geometrical forms has been recast in the kindergarten grammars as vocabularies of shapes and a system of shape rules that are specified in terms of spatial relations between shapes in the vocabularies (Knight, 1994; Stiny, 1980). The emphasis in the kindergarten grammars is placed on the formulation of the shape rules and the degree of precision and control in their use; the specific vocabularies of shapes used in the grammars are chosen simply to make clear the pedagogical value of the kindergarten method.

The symmetries of the three-dimensional world play a significant role in this spatial system. The role of two-dimensional symmetry in the shape grammar formalism has been addressed recently (Knight, 1995; Stiny, 1991). A restricted first view is taken here for a specific class of symmetries of the three-dimensional world so that the designs created by these shape rules may be correlated with the symmetry properties under which the rules apply. The three-dimensional Froebel gifts 3–6 are chosen for the exposition of this material; the 6 individual solids that produce the 4 building gifts are shown in figure 1. Other vocabularies of shapes such as the *Bauhaus blocks*, *Lowenfeld's poleidoblocks*, *Froebel's tetrahedral gift*, or the *Tangram blocks*, might also be used (Schroder, 1992). Nevertheless, as they are produced by various dissections of the cube and described by symmetry groups intricately related in successive nestings, Froebel's gifts are paradigmatic for this kind of analysis. A detailed account of the symmetry groups of the building blocks is valuable in the first pedagogical stages of the grammatical formalism and in making reasoned choices between a multitude of possible spatial relations and labeling schemes.

Figure 1. The 6 solids in Froebel gifts 3–6.

2 The symmetry groups of the Froebel gifts 3–6

A *symmetry* of a shape is a transformation, such as a rotation, rotor reflection, or any other isometric transformation, which leaves the shape invariant. The collection of all such transformations is the *symmetry group* of a given figure. Symmetry groups are classified according to their translational structure and the dimensionality of the space that contains their elements. All of the symmetry groups are subgroups of the *Euclidean group E* consisting of all isometries, which is a subgroup of the *similarities group S* consisting of all similarities. The 10 symmetry groups G_{ij} in Euclidean space, where i is the number of axes of translation, and j is the dimension of space, are given in table 1.

The symmetry groups of the 6 Froebel building blocks are necessarily subgroups of the three-dimensional point group G_{03} In three-dimensional space there are altogether 14 types of three-dimensional point groups: 7 finite *polyhedral groups* and 7 infinite sets of *prismatic groups* (Lockwood and Macmillan, 1978). The 7 polyhedral groups are the symmetry groups of the 5 regular platonic polyhedra. The 7 prismatic groups are the symmetry groups of an infinite class of polyhedra which permit n-fold rotations or n-fold rotor reflections about one axis, whereas any other rotations or rotor reflections in the symmetry group are necessarily about a two-fold axis (Yale, 1968). The symmetry group of the cube is an instance of 1 polyhedral type, the complete *octahedral* group, and the symmetry groups of the 5 other Froebel solids are instances of 2 infinite types of the prismatic symmetry groups.

Table 1. Labeling schemes of the 10 symmetry groups G_{ij}.

Groups				
0-dimensional	1-dimensional	2-dimensional	3-dimensional	
G_{00}	G_{01}	G_{02}	G_{03}	point groups
	G_{11}	G_{12}	G_{13}	line groups
		G_{22}	G_{23}	plane groups
			G_{33}	space groups

The order of symmetry of each group is the number of symmetry elements that each group contains. The order of symmetry of the cube is 48, the order of the pillar and the square is 16, the order of the oblong is 8, and, lastly, the order of the half-cube and the quarter-cube is 4.

The minimum number of isometries that can generate the symmetries of the shapes as well as the minimum sets of such isometries can be specified uniquely for each shape (Shubnikov and Koptsik, 1974). These isometries, the so-called *generators* for each group, differ for each shape. The symmetry groups of the pillar, the square and the oblong, require 3 generators, whereas the symmetry groups of the cube, the half-cube and the quarter-cube, require 2 generators. The minimum sets of generators for symmetry groups differ as well (Schattschneider, 1986). There are 16 different sets for the groups of the pillar, the square, and the oblong, but only 2 sets for the symmetry groups of the half-cube and the quarter-cube.

There is no standard notational scheme for symmetry groups. A nice correlation of several types of notational schemes for three-dimensional symmetry groups is given in Cromwell (1997). The one used in this paper is the so-called *noncoordinate notation* for symmetry classes (Shubnikov and Koptsik, 1974). In this notation, n signifies the number of rotations around an axis of order n, m signifies a reflection about a mirror plane, and $2\tilde{n}$ signifies a rotor reflection about a mirror-rotation axis $\tilde{n}$ which is necessarily of an even order. The notation, order of symmetry, and the number of the generators for the solid Froebel blocks are given in table 2 (see over).

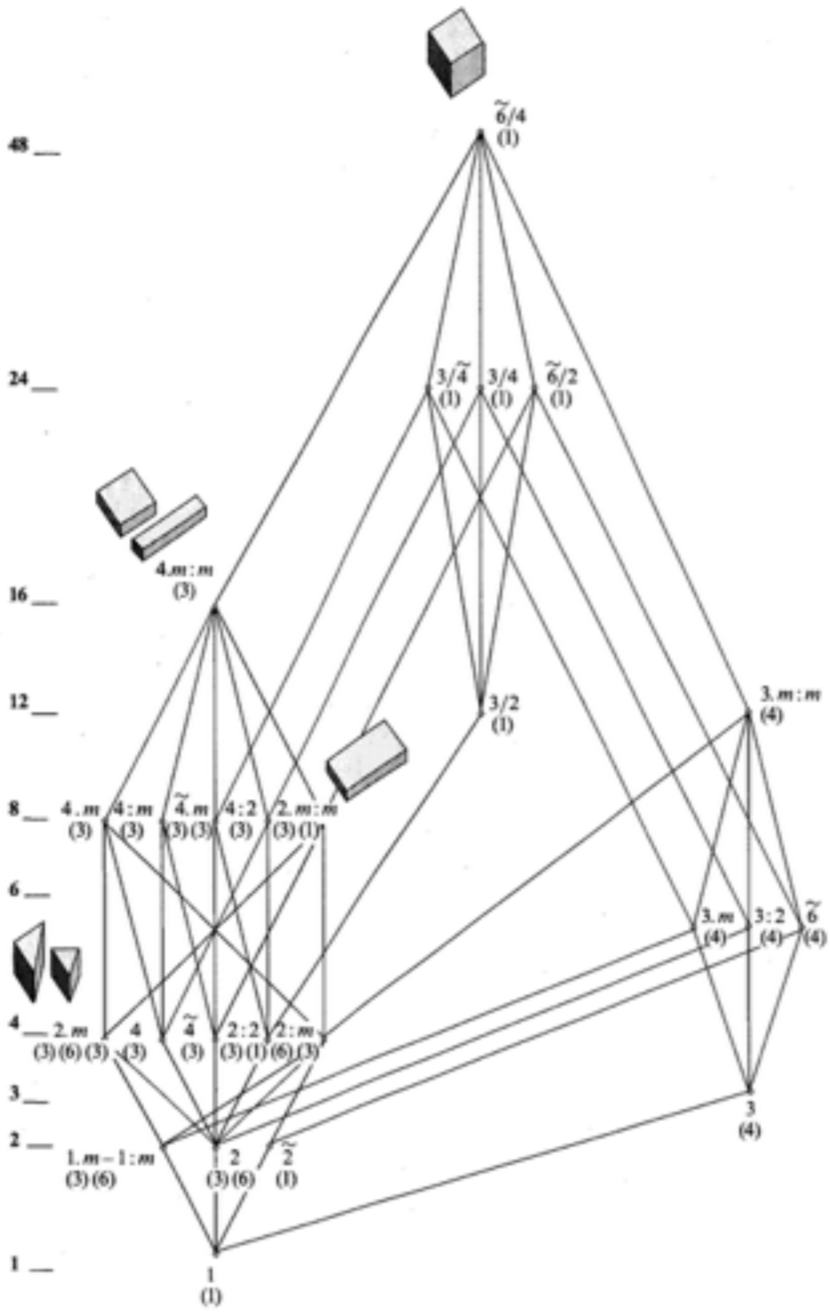


Figure 2. Subgroup relations for the three-dimensional point symmetry groups of the Froebel blocks.

The Froebel blocks are all produced by specific perpendicular and diagonal bisections of the cube and all exhibit various amounts of symmetry. This geometric observation leads to the algebraic statement that these groups are related to one another and all together with the symmetry group of the cube. The relation of the 4 symmetry groups of the Froebel building blocks with the symmetry group of the cube is nicely captured in a diagram illustrating the subgroup relations within the full

Table 2. Classification of the Froebel blocks in terms of their symmetry groups.

				
Noncoordinate notation	$\tilde{6}/4$	$4.m:m$	$2.m:m$	$2.m$
Order of symmetry	48	16	8	4
Group generators	2	3	3	2

symmetry group of the cube in figure 2. All symmetry subgroups are nested the one within the other. The lattice of the symmetry subgroups of the half-cube and the quarter-cube is contained within the lattice of the symmetry subgroups of the oblong. Both lattices are contained within the lattice of the symmetry subgroups of the pillar and the square and all of them are contained within the lattice illustrating the partial order of the subgroups of the octahedral group.

3 The conjugacy classes of the Froebel gifts 3 – 6

A finer exploration of the structure of the building gifts takes into account the geometrical nature of each isometry with respect to each individual spatial structure. This analysis is based on the analytical method employed for the partition, say, of the Euclidean group E, consisting of all isometries, into 10 groups G_{ij}, or for the partition of the three-dimensional point group G_{03}, consisting of all rotations, reflections, and rotor reflections about a point O, into 7 polyhedral groups and 7 infinite sets of prismatic groups. This finer classification distinguishes between these symmetries, so far considered typologically the same, with respect to the effect that they impose on each individual spatial structure. In these corresponding spaces, neither are all reflections the same, nor are all two-fold rotations of the same type. In the cube, for example, a reflection in a face plane interchanges all 3 pairs of faces, whereas a reflection in an edge plane interchanges 2 faces and leaves 2 faces still. Both isometries

Table 3. Conjugacy classes of the symmetry elements of the Froebel blocks.

					
I	1	1	1	1	Identity
A	6	2	-	-	90° rotation – face axis
B	3	3	3	1	180° rotation – face axis
C	6	2	-	-	180° rotation – edge axis
D	8	-	-	-	120° rotation – vertex axis
E	3	3	3	2	Reflection in face plane
F	6	2	-	-	Reflection in edge plane
H	8	-	-	-	120° rotor inversion – vertex
J	6	2	-	-	90° rotor inversion – face
K	1	1	1	-	Inversion in centroid
	48	16	8	4	*Isometries*
	10	8	4	3	*Conjugacy classes*

Table 4. Conjugacy classes of the symmetry subgroups of the Froebel blocks.

					*Class structure*
I	1	1	1	1	I
A	3	1	-	-	$I+B+2A$
B	3	3	3	1	$I+B$
C	6	2	-	-	$I+C$
D	4	-	-	-	$I+2D$
E	3	3	3	2	$I+E$
F	6	2	-	-	$I+F$
H	4	-	-	-	$I+K+2D+2H$
J	3	1	-	-	$I+B+2J$
K	1	1	1	-	$I+K$
BB	1	1	1	-	$I+3B$
BC	3	1	-	-	$I+B+2C$
BE	3	3	3	-	$I+B+E+K$
BF	3	1	-	-	$I+B+2F$
CE	6	2	-	-	$I+C+E+F$
CK	6	2	-	-	$I+C+F+K$
EE	3	3	3	1	$I+B+2E$
CD	4	-	-	-	$I+2D+3C$
FF	4	-	-	-	$I+2D+3F$
AB	3	1	-	-	$I+2A+2C+3B$
AE	3	1	-	-	$I+B+E+K+2A+2J$
BFF	3	1	-	-	$I+2F+2J+3B$
CJ	3	1	-	-	$I+B+2C+2E+2J$
EEE	1	1	1	-	$I+K+3B+3E$
EF	3	1	-	-	$I+B+2A+2E+2F$
EFF	3	1	-	-	$I+B+E+K+2C+2F$
BD	1	-	-	-	$I+3B+8D$
CF	4	-	-	-	$I+K+2D+2H+3C+3F$
BBC	3	1	-	-	$I+K+2A+2C+2F$ $+2J+3B+3E$
CCC	1	-	-	-	$I+3B+6F+6J+8D$
DEE	1	-	-	-	$I+K+3B+3E+8D+8H$
R	1	-	-	-	$I+3B+6A+6C+8D$
G	1	-	-	-	$I+K+3B+3E+6A+6C$ $+6F+6J+8D+8H$
	98	35	16	5	*Subgroups*
	33	23	8	4	*Conjugate classes*

permute the faces in a different way and they are thus classified as different. The relation that partitions the sets of symmetry elements of a group into equivalent classes of isometries that are characterized by the same type is called a *conjugacy* relation. Conjugacy is an *equivalence* relation between elements. The choice of the spatial system within which the conjugacy relations are defined affects the ways that a designer looks at a design. Different conjugacy relations result in different decompositions and different orderings of resulting subgroups (March, 1983; 1996).

Formally, given elements x, y of a group G, x is conjugate to y if $g^{-1}xg = y$ for some $g \in G$. The equivalence classes are called *conjugacy classes* and the elements within the same class must have the same order. The conjugacy class of an element x in G is found by calculating $g^{-1}xg$ for every $g \in G$. Similarly the conjugacy class of a

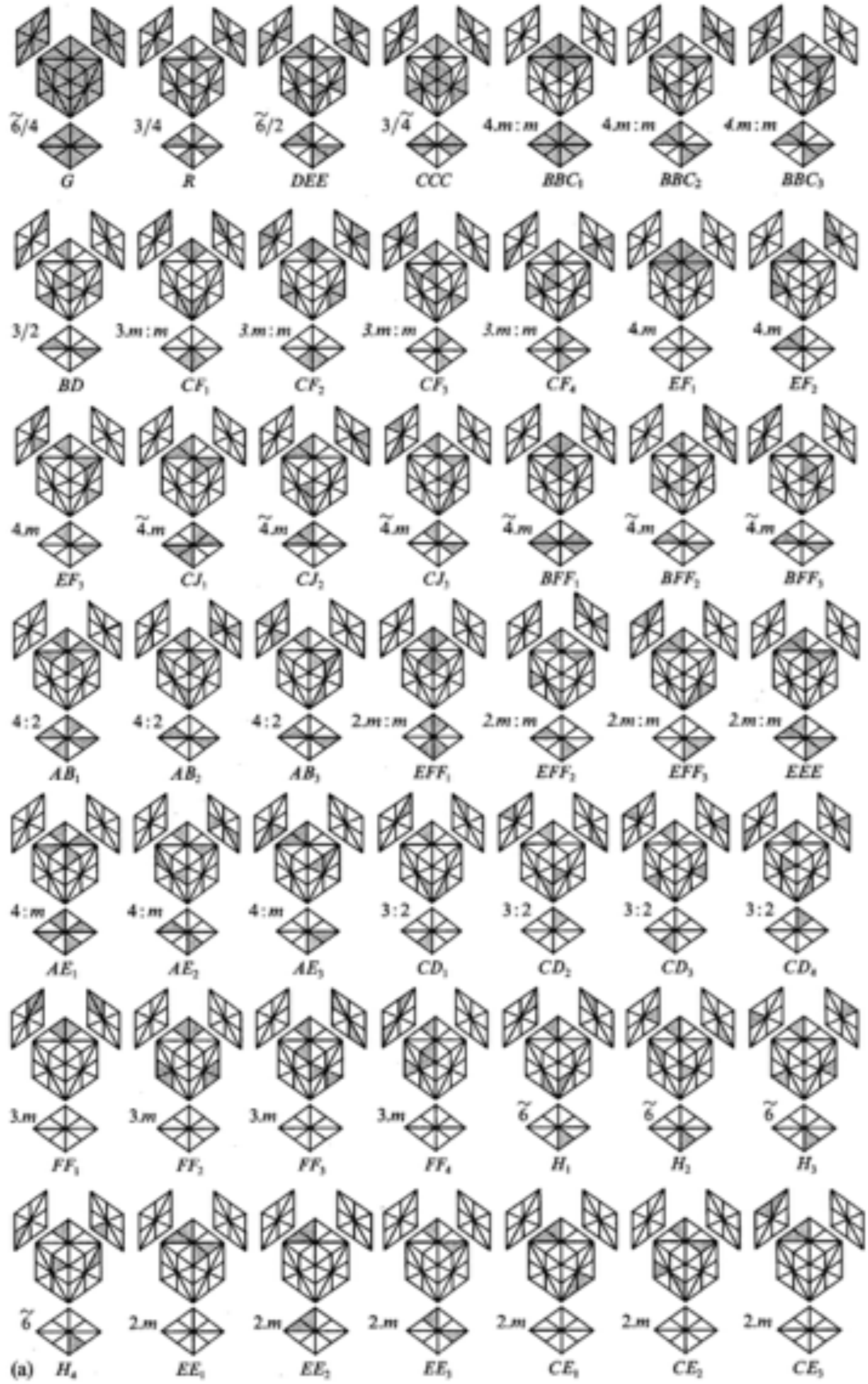
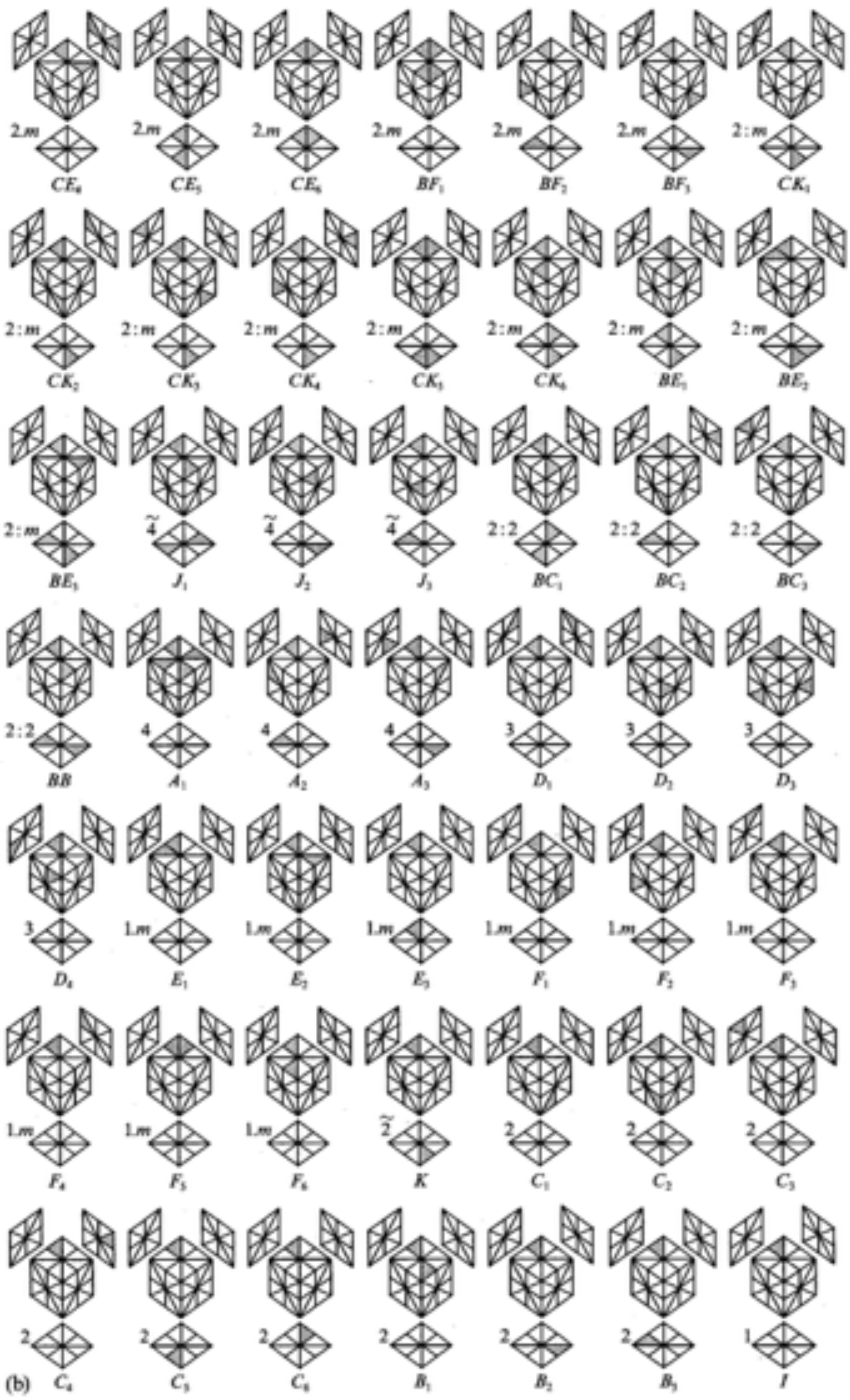


Figure 3. Catalogue of all 98 subgroups of the symmetry group of the cube.



(b)

Figure 3 (continued).

power of x, say x^m, is found by calculating $g^{-1}x^m g$ for every $g \in G$. Once some of the conjugacy classes have been found, the others are easily calculated (Armstrong, 1988).

The 48 elements of the symmetry group of the cube naturally split into 10 conjugacy classes (Yale, 1968, pages 12–13). The conjugacy classes of the symmetries of the other 5 Froebel blocks are subsets of these 10 classes. The 16 elements of the pillar and the square are partitioned into 8 classes, the 8 elements of the oblong into 4 classes, and the 4 elements of the half-cube and the quarter-cube into 3 classes. The explicit descriptions of these types of symmetries and their order for the 6 Froebel blocks are given in table 3. These conjugacy classes of symmetries combine to create conjugacy groups within the symmetry groups of each individual solid. The conjugate subgroups show essentially the different types of symmetries within each block. There are 33 conjugacy classes in the octahedral group (Lunnon, 1972) and it can easily be deduced that the 3 types of symmetry groups of the building blocks can be further partitioned into 23, 8, and 4 conjugacy classes, respectively. The explicit descriptions of these classes of symmetry subgroups are given in table 4.

The symmetry groups can be even further scrutinized in terms of all the distinct subgroups that are formed out of all possible combinations of the elements of the group. The enumeration of all subgroups for a given group is a very difficult task and has been carried through only for selected few finite groups (Budden, 1972). The octahedral group has 98 subgroups which all together form the 33 conjugacy classes of the cube (Lunnon, 1972). All of the subgroups of the other Froebel blocks are contained within these 98 groups. It can easily be checked that the symmetry group of the pillar contains 35 subgroups, the symmetry group of the oblong contains 16 subgroups, and the symmetry group of the half-cube contains 4 subgroups. The subgroups of the cube are shown in figure 3. All the symmetry subgroups of the other Froebel blocks can be deduced by correlating the results of table 3 and table 4 with the illustrations of all the subgroups in figure 3. Partial orderings of the subsymmetries of the blocks can be drawn by correlating the lattice in figure 2 with the catalogue of all the subgroups in figure 3.

4 The cycle representation of the Froebel gifts 3–6

The exploration of the symmetry properties of the Froebel building gifts is further advanced through the use of permutations and the isomorphic representation of the symmetry groups of the blocks with corresponding permutation groups (Baglivo and Graver, 1976; March, 1993). Any symmetry transformation can be considered as a specific permutation of a set that leaves the structure under consideration invariant (Yale, 1968). The decompositions of a three-dimensional shape into its basic spatial elements—points, lines, and planes—are the best members for this set. For example, in the case of the cube, the elements of the set that are permuted by the symmetry transformations can be its 8 vertices, its 12 edges, or its 6 faces. Vertices, edges, and faces are not the only candidates for this set. In fact, a description of a three-dimensional geometrical figure in terms of its internal three-dimensional diagonals is considered by mathematicians as the most elegant and parsimonious one because it has the simplest structure; it involves a minimum set of elements which specify completely the properties of the structure under consideration. All descriptions of the structure of the oblong are given in figure 4.

The elements of the symmetry groups of the Froebel building gifts can be written down in the form of cycles of permutations of a set consisting of the vertices, edges, faces, or diagonals of each block. The sum of all combinations or products of cycles divided by the total number of the elements in the group is the *cycle index* of the

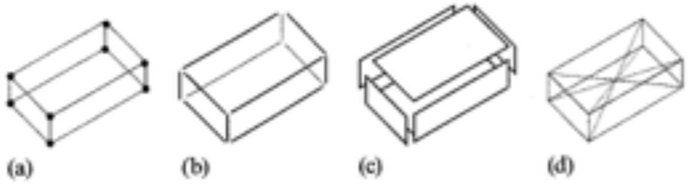


Figure 4. Description of an oblong in terms of its vertices, edges, faces, and internal diagonals.

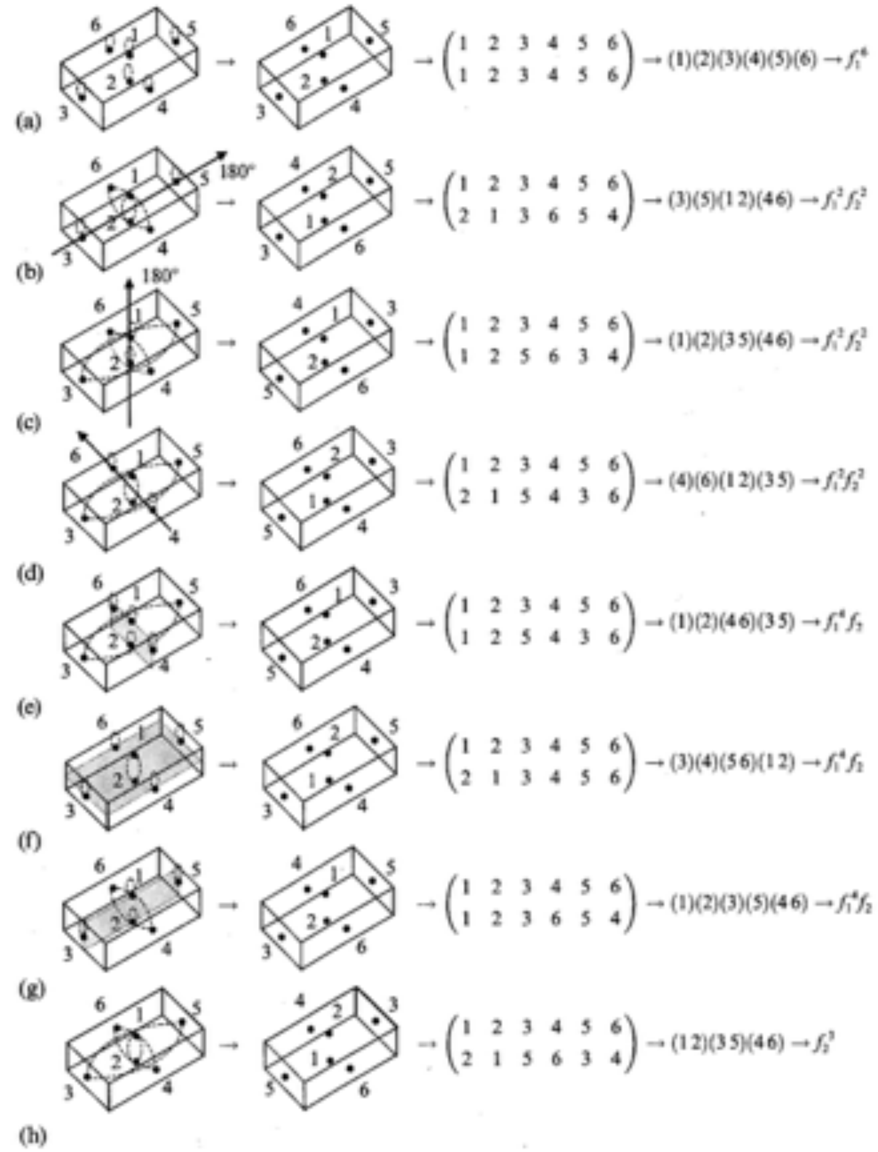


Figure 5. The 8 permutations of the set of faces of the oblong.

Table 5. The complete cycle indices of the permutation groups of the vertices, edges, and faces of the Froebel blocks.

$C_V = \frac{1}{24}(f_1^8 + 9f_2^4 + 6f_4^2 + 8f_1^2 f_3^2)$
$C_V = \frac{1}{48}(f_1^8 + 13f_2^4 + 12f_4^2 + 8f_1^2 f_3^2 + 8f_2 f_6 + 6f_1^4 f_2^2)$
$C_E = \frac{1}{24}(f_1^{12} + 3f_2^6 + 8f_3^4 + 6f_4^3 + 6f_1^2 f_2^5)$
$C_E = \frac{1}{48}(f_1^{12} + 4f_2^6 + 8f_3^4 + 12f_4^3 + 8f_6^2 + 12f_1^2 f_2^5 + 3f_1^4 f_2^4)$
$C_F = \frac{1}{24}(f_1^6 + 6f_2^3 + 8f_3^2 + 6f_1^2 f_4 + 3f_1^2 f_2^2)$
$C_F = \frac{1}{48}(f_1^6 + 7f_2^3 + 8f_3^2 + 8f_6 + 9f_1^2 f_2^2 + 6f_1^2 f_4 + 3f_1^4 f_2 + 6f_2 f_4)$

$C_V = \frac{1}{8}(f_1^8 + 5f_2^4 + 2f_4^2)$
$C_V = \frac{1}{16}(f_1^8 + 9f_2^4 + 4f_4^2 + 4f_1^4 f_2^2)$
$C_E = \frac{1}{8}(f_1^{12} + 3f_2^6 + 2f_4^3 + 2f_1^2 f_2^5)$
$C_E = \frac{1}{16}(f_1^{12} + 4f_2^6 + 4f_4^3 + 4f_1^2 f_2^5 + 3f_1^4 f_2^4)$
$C_F = \frac{1}{8}(f_1^6 + 2f_2^3 + 2f_1^2 f_4 + 3f_1^2 f_2^2)$
$C_F = \frac{1}{16}(f_1^6 + 3f_2^3 + 5f_1^2 f_2^2 + 2f_1^2 f_4 + 3f_1^4 f_2 + 2f_2 f_4)$

$C_V = \frac{1}{4}(f_1^8 + 3f_2^4)$
$C_V = \frac{1}{8}(f_1^8 + 7f_2^4)$
$C_E = \frac{1}{4}(f_1^{12} + 3f_2^6)$
$C_E = \frac{1}{8}(f_1^{12} + 4f_2^6 + 3f_1^4 f_2^4)$
$C_F = \frac{1}{4}(f_1^6 + 3f_1^2 f_2^2)$
$C_F = \frac{1}{8}(f_1^6 + f_2^3 + 3f_1^2 f_2^2 + 3f_1^4 f_2)$

$C_V = \frac{1}{2}(f_1^5 + f_1 f_2^2)$
$C_V = \frac{1}{4}(f_1^5 + 2f_1^3 f_2 + f_1 f_2^2)$
$C_E = \frac{1}{2}(f_1^9 + f_1 f_2^4)$
$C_E = \frac{1}{4}(f_1^9 + f_1 f_2^4 + 2f_1^3 f_2^3)$
$C_F = \frac{1}{2}(f_1^5 + f_1 f_2^2)$
$C_F = \frac{1}{4}(f_1^5 + 2f_1^3 f_2 + f_1 f_2^2)$

corresponding permutation group. The computation of the cycle index of the permutation group of the faces of the oblong is shown in figure 5.

The 8 permutations that leave the structure of the oblong invariant are isomorphic to the 8 symmetries of the shape and naturally split into 4 equivalent conjugacy classes. First is the permutation corresponding to the rotation consisting of no motion at all; the identity permutation depicted in figure 5(a). This permutation has 6 cycles of order 1 and is represented as f_1^6. The next 3 permutations correspond to 3 half-turns about the 3 face axes and are given in figures 5(b), 5(c), and 5(d). Each permutation has 2 cycles of order 1, and 2 cycles of order 2 and is represented as $f_1^2 f_2^2$. The next 3 permutations correspond to reflections about the 3 mirror face planes and are given in figures 5(e), 5(f), and 5(g). Each permutation has 4 cycles of order 1, and 1 cycle of order 2 and is represented as $f_1^4 f_2$. The last permutation corresponds to a central inversion, that is, a reflection through a point, in this case through the centroid of the oblong, and is given in figure 5(h). This permutation has 3 cycles of order 2 and contributes the term f_2^3. The cycle index of the permutation group of the faces of the oblong is taken in a straightforward way on dividing the sum of all the k-cycles of order k and their products by the number of the elements of the symmetry group of the oblong. The cycle index induced by proper isometries is denoted by the symbol C_F, and the cycle index induced by proper and improper isometries is denoted by the symbol C_F. The complete list of the cycle indices of the permutation groups of the vertices, edges, and faces of the 6 Froebel blocks is given in table 5.

5 The colored symmetry of the Froebel gifts 3 – 6

The symbolic sentences that describe the structure of the Froebel blocks can be used in the exploration of the *polychromatic* or *weighted symmetry* of the blocks. The appropriate method for this inquiry has been given by Polya in his theory on counting nonequivalent configurations with respect to a given permutation group (Polya et al, 1983). Essentially, Polya's theory of counting specifies the numbers of different ways in which k qualities can be assigned to n vertices of an n-cornered figure without considering any two arrangements as different if they can be transformed one to another by a symmetry operation. Polya's formalism provides the answer even if $k > n$; here, in other words, it is possible to compute the perfect colorings or weighted symmetries of the blocks for any number of colors.

The thrust of Polya's theorem in an ingenious substitution of the cycles of permutations with a figure inventory of elements that are to be permuted upon the structure. For f_k, a cycle of order k, and a figure inventory of elements, x, y, ..., w, this substitution is given in equation (1):

$$f_k = x^k + y^k + ... + w^k . \tag{1}$$

The resulting coefficients of the terms after the expansion of the cycle index in powers of x, y, ..., w give the nonequivalent configurations of the elements x, y, ..., w upon the structure. In our case, the coefficient of a term, say, $x^r y^s z^t$, gives the distinct color configurations of r x-color faces, s y-color faces, and t z-color faces of a Froebel solid block.

The computation of the coloring schemes up to 3 colors for the oblong is given below. The complete cycle index of the permutation group of the faces of the oblong under proper and improper isometries is given in expression (2):

$$\underline{C}_f = \tfrac{1}{8}\left(f_1^6 + 3f_1^2 f_2^2 + 3f_1^4 f_2 + f_2^3 \right) . \tag{2}$$

The substitution of a figure inventory $(x + y + z)$ of 3 colors x, y, z in expression (2) according to expression (1) produces the symbolic sentence (3):

$$\begin{aligned} \underline{C}_f = \tfrac{1}{8}\big[&(x+y+z)^6 + 3(x+y+z)^2(x^2+y^2+z^2)^2 + 3(x+y+z)^4(x^2+y^2+z^2) \\ &+ (x^2+y^2+z^2)^3 \big] . \end{aligned} \tag{3}$$

The coefficients of each entry in sentence (3) can be taken in a straightforward way by the *multinomial theorem* (4):

$$(x+y+z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t . \tag{4}$$

The coefficients of all the terms in sentence (3) can be found only after the expansion of the symbolic sentence. There is no need to compute the total sentence because of its inherent symmetry. The first half of its 3 identical parts is given in sentence (5):

$$\underline{C}_f = 1x^6y^0z^0 + 3x^5y^1z^0 + 6x^4y^2z^0 + 7x^3y^3z^0 + 9x^4y^1z^1 + 15x^3y^2z^1 + 21x^2y^2z^2 . \tag{5}$$

Table 6. Enumeration of all n-coloring schemes of the Froebel blocks, for $n \leqslant 3$.

								
	$\sum/24$	$\sum/48$	$\sum/8$	$\sum/16$	$\sum/4$	$\sum/8$	$\sum/2$	$\sum/4$
x^6	1	1	1	1	1	1		
x^5y	1	1	1	1	3	3		
x^4y^2	2	2	2	4	6	6		
x^3y^3	2	2	2	4	8	7		
x^4yz	2	2	5	5	9	9		
x^3y^2z	3	3	18	18	18	15		
$x^2y^2z^2$	6	5	15	12	27	21		
x^5							1	1
x^4y							3	3
x^3y^2							6	5
x^3yz							10	8
x^2y^2z							16	11

The corresponding coefficients of the terms of sentence (5) show the nonequivalent color configurations of an oblong colored with a maximum of 3 colors. The results are quite succinct: for example, the term $21x^2y^2z^2$ signifies that there are only 21 nonequivalent configurations of an oblong with 2 x-color faces, 2 y-color faces, and 2 z-color faces. Table 6 summarizes the results for both proper and complete isometries. The analytical computation for all coloring schemes up to 3 colors for all 6 Froebel building blocks is given elsewhere (Economou, 1998a). The complete catalogue of all nonequivalent color configurations for the category 3 x-color faces, 2 y-color faces, and 1 z-color face is given in figure 6.

6 Discussion

There are many advantages of the application of Polya's theorem in architectural research. First, the enumeration of nonequivalent configurations applies to many similar classes of designs. The cycle indices of the permutation groups of the edges and faces of the Froebel blocks are identical to the corresponding cycle indices of other similar classes of three-dimensional shapes. This occurs because the permuted elements of a set, irrespective of whether they are vertices, edges, or faces are discrete; lines and faces are computed as points. Three classes of n-cornered shapes based on the structure of the oblong are given in figure 7.

Second, the enumeration of nonequivalent configurations applies to any coloring schema chosen by a designer. Any rule may be defined that associates a basic element of the blocks with a coloring scheme. At the very least, a point may denote color in specific proportionate parts of 3 adjacent faces, a line may denote color in specific proportionate parts of 2 adjacent faces, and a diagonal, if used, may denote color in 2 diagonal portions of the blocks in its direction. Note that, in this last case, the coloring scheme is a subset of the analogous scheme related with the vertices. These 4 coloring schemes associated with the vertices, edges, faces, and diagonals of the blocks are given in figure 8 (see over).

Third, the theorem specifies the numbers of enantiomorphs in all classes; the difference between the 2 types of nonequivalent configurations based on rotational and complete symmetries denotes the number of enantiomorphic colorings, that is,

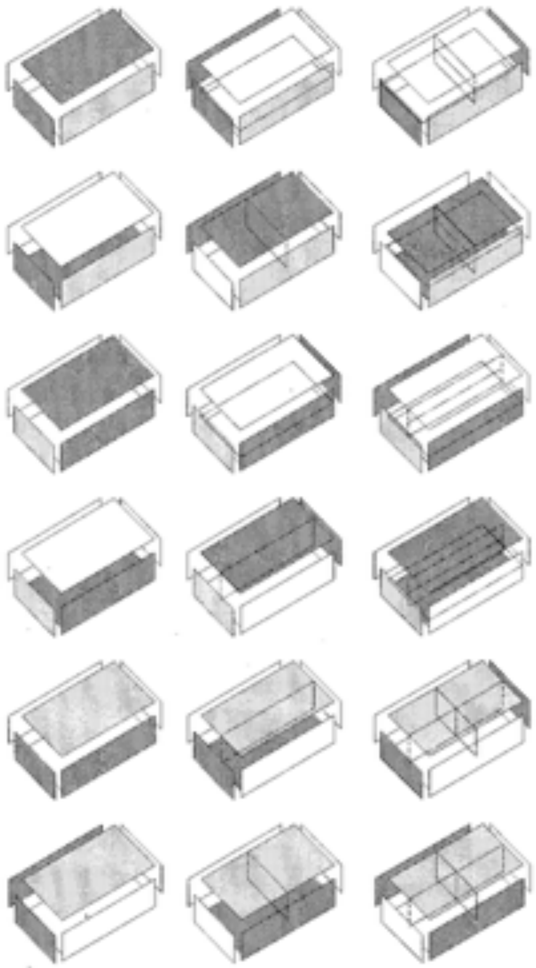


Figure 6. Enumeration of all nonequivalent configurations of 3 x-color faces, 2 y-color faces, and 1 z-color face for the oblong.

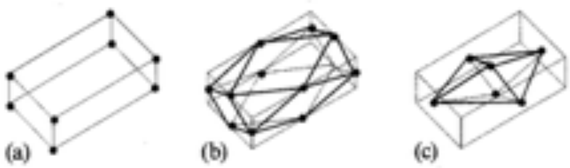


Figure 7. Classes of shapes based on the structure of the oblong.

the number of the coloring schemes that come in handed versions. For example, in table 6 the existence of 6 rotational coloring schemes and 5 reflectional coloring schemes for the cube in the category 2 : 2 : 2 denotes that 2 of the rotational coloring schemes are enantiomorphic versions of each other. In other words, there is 1 enantiomorphic pair of cubes weighted with 2 x-color faces, 2 y-color faces, and 2 z-color faces. Analogous results can be drawn for all the other Froebel blocks for every coloring scheme.

The existence of enantiomorphic forms in the various classes gives additionally some other types of information with respect to the symmetry properties of the blocks.

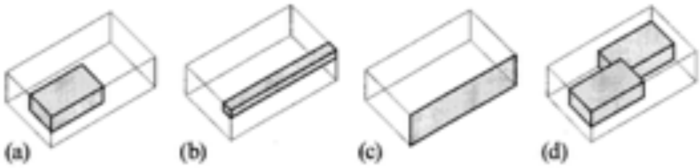


Figure 8. Alternative coloring schemes of the oblong.

First, the coloring schemes that break down the symmetry to 1 can be determined, provided that each face is painted with 1 color. For example, the nonexistence of enantiomorphic colored oblongs in the configuration 4:1:1 means that 3 colors in this combination cannot reduce the symmetry of this shape to 1. If the arrangements 3:2:1 and 2:2:2 are used, the symmetry collapses to 1 in 3 and 6 ways, respectively. Second, the minimum number of colors that break down the symmetry of these shapes to 1 can be determined. For example, the existence of an enantiomorphic colored oblong in the configuration 3:3 means that 2 colors in this combination can reduce the symmetry of this shape to 1. The corresponding minimum numbers of colors that break down the symmetry to 1 for all 6 Froebel blocks are: 2 for the half-cube and the quarter-cube in the ratio 3:2; 2 for the oblong in the ratio 3:3; 3 for the pillar and the square in the proportion 2:2:2; and 3 for the cube in the proportion 2:2:2. These colorings are illustrated in figure 9.

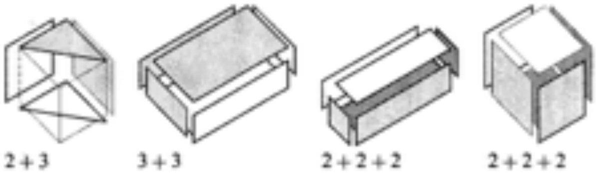


Figure 9. Enantiomorphic coloring schemes of the Froebel blocks.

And last, but not least, the theorem in itself provides the information for its generalization of counting nonequivalent configurations of k elements to n clusters of entities on n vertices of an n-cornered figure! Figure inventory in the standard format of the theorem is the sum of variables or choices in a design context. The generalization of the theorem is attained through a more general form of the figure inventory. The idea is that each choice in the figure inventory is associated with a value that is not a unique variable. In the previous example, the figure inventory consists of 3 colors and is denoted as $x + y + z$. The assignment of these colors to the faces of each block is equivalent to the arrangement of 6 colored beads at the vertices of the dual three-dimensional figure for each block (figure 7). If we had beads of 3 different colors x, y, z, and we wanted to arrange them in clusters of 3s at the vertices of this dual figure and wanted to know in how many different ways we can place, say, 3 x-color beads together in a vertex, 2 x-color beads and 1 y-color bead in a vertex, and last, 1 x-color bead, 1 y-color bead, and 1 z-color bead in a vertex, then these possibilities would be represented in the figure inventory as $x^3 + x^2y + xyz$. If we substitute this figure inventory in the cycle index of an n-cornered figure, the resulting coefficient of $x^ry^sz^t$ will be the number of placing clusters such that exactly r x-color beads, s y-color beads, and t z-color beads are used! For example, in the case of the oblong under these conditions, the coefficient of a term such as $x^{13}y^3z^3$ in the expansion will tell us the number of clusters such that 13 x-color beads, *3* y-color beads, and 2 z-color beads are used. The general form of the figure inventory is defined then for 3 variables in equation (6) whereas a_{rst} is the number of figures with r x-color beads, s y-color beads, and t z-color beads:

$$f(x,y,z) = \sum_{r,s,t=0}^{\infty} a_{rst} x^r y^s z^t . \quad (6)$$

The extension of the theorem to k variables can be determined (Polya et al, 1983, page 74–85).

5 Conclusion

The use of three-dimensional symmetry in the analysis of form is the focus of this paper. To this extent, the symmetry properties of the Froebel building gifts have been examined in detail. I have focused here on the enumeration of the symmetry elements of the Froebel blocks, the identification of *equivalence* or *conjugacy* classes among these elements and their symmetry subgroups, and the computation of the cycle index of the permutation group of the vertices, edges, and faces for each individual block. The power of Polya's theorem in the analysis of form was also established. A variety of different domains where these formal tools may be applied is readily available: in urban design, configurations based on a gridiron pattern can be enumerated; in color grammatical formalism, the number of basic color grammars can be extended to encompass all permutations of colors in the initial shape rules; in music, all scalar and chordal patterns based on the twelve-tone equal temperament scale can be identified; in color theory, all color patterns based on a model, for example, the model of the cube-octahedron, can be classified (Economou, 1998a; 1998b). Symmetry is intimately related to *compositional development*, and its three-dimensional version provides an inexhaustible source of inspiration for the designer and the researcher alike.

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